Name Doa'a Nassar, ID 43631 56 Excellent

MTH 205, Differential Equations , Exam One

Ayman Badawi

QUESTION 1. (28 points)

(i) $\ell\{(x+2)^2\} = \ell\{x^2 + 4x + 4\}$

$= \frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}$

(ii) $\ell\{(x-2)(e^x+1)\} = \ell\{xe^x + x - 2e^x - 2\}$

$= \ell\{xe^x\} + \ell\{x\} - 2\ell\{e^x\} - 2\ell\{1\}$

$= \frac{1}{(s-1)^2} + \frac{1}{s^2} - \frac{2}{s-1} - \frac{2}{s}$

$\ell\{3x^2\} = \frac{2\cdot 3}{s^3} = \frac{6}{s^3}$

(iii) $\ell\{\int_0^x 3r^2e^{-2x-3r}\ dr\}$

$= \ell\{\int_0^x 3r^2\, e^{-2(x-\frac{3}{2}r)}\, dr\}$ $-5r$

7 $= \ell\{\int_0^x 3r^2\, e^{-2(x-r)} \cdot e^{-r}\}$

$= \ell\{e^{-x}3x^2 * e^{-2x}\} = \frac{6}{(s+1)^3} \cdot \frac{1}{s+2}$

(iv) $\ell^{-1}\{\frac{s+6}{s^2+4s+20}\}$ $+4-4$

$\ell^{-1}\left\{\frac{s+6\ (+2-2)}{(s+2)^2+16}\right\} = \ell^{-1}\left\{\frac{(s+2)+4}{(s+2)^2+16}\right\} = e^{-2x}(\cos 4x + \sin 4x)$

$* \ \ell^{-1}\left\{\frac{s}{s^2+16}\right\} + \ell^{-1}\left\{\frac{4}{s^2+16}\right\} = \cos 4x + \sin 4x$

(v) $\ell^{-1}\{\frac{e^{-2s}}{s^2+6s-7}\}$

$\ell^{-1}\left\{\frac{e^{-2s}}{s^2+6s-7}\right\} = U(x-2)\, f(x-2)$

$\frac{1}{s^2+6s-7} = \frac{1}{(s+7)(s-1)} = \frac{A}{s+7} + \frac{B}{s-1}$

$s=-7 \quad s=1$

$A = -\frac{1}{8} \quad B = \frac{1}{8}$

$\ell^{-1}\{F(s)\} = f(x) = \ell^{-1}\left\{\frac{1}{s^2+6s-7}\right\} = \ell^{-1}\left\{\frac{-1}{8(s+7)} + \frac{1}{8(s-1)}\right\}$

$= -\frac{1}{8}e^{-7x} + \frac{1}{8}e^{x}$

$\ell^{-1}\left\{\frac{e^{-2s}}{s^2+6s-7}\right\} = U(x-2)\left(\frac{-1}{8}e^{-7(x-2)} + \frac{1}{8}e^{x-2}\right)$

(vi) Calculate $\int_0^\pi cos(2x)e^{-3x}\ dx$ [Hint : maybe you want to use the fact that Co(2x) has period equals to π and you know $\ell\{cos(2x)\}$].

$$\ell\{\cos(2x)\} = \frac{1}{1-e^{-\pi s}} \int_0^\pi e^{-sx} \cdot f(x)\,dx$$

$$= \frac{1}{1-e^{-\pi s}} \cdot \int_0^\pi e^{-sx} \cos 2x\,dx$$

$$\ell\{\cos 2x\} = \frac{s}{s^2+4}$$

$$\int_0^\pi e^{-sx}\cos 2x\,dx = \frac{s(1-e^{-\pi s})}{s^2+4}$$

as $s \to -3$: $\int_0^\pi \cos 2x\, e^{-3x}\,dx = \boxed{\frac{-3(1-e^{3\pi})}{13}}$

(vii) $\ell^{-1}\{\frac{s+3}{(s+7)^3}\}$

$$\ell^{-1}\left\{\frac{s+7-4}{(s+7)^3}\right\}$$

$$* \ \ell^{-1}\left\{\frac{s-4}{s^3}\right\} = \ell^{-1}\left\{\frac{1}{s^2}\right\} - \frac{4}{2}\ell\left\{\frac{1 \cdot 2}{s^3}\right\}$$

$$= x - 2x^2$$

$$\ell^{-1}\left\{\frac{s+3}{(s+7)^3}\right\} = (x-2x^2)e^{-7x}$$

$\cos 2x * y(x)$

QUESTION 2. (10 points) Solve for $y(x)$: Given $y'(x) = sin(2x) + 8\int_0^x cos(2r)y(x-r)\ dr$, where $y(0) = 0$.

$$sY(s) - y(0) = \frac{2}{s^2+4} + 8\,\ell\{\cos 2x * y(x)\}$$

$$sY(s) = \frac{2}{s^2+4} + 8\left(\frac{s}{s^2+4}\right)(Y(s))$$

$$\frac{(s^2+4)}{(s^2+4)} sY(s) - \frac{8s}{s^2+4} Y(s) = \frac{2}{s^2+4}$$

$$\frac{s^3-4s}{s^2+4} Y(s) = \frac{2}{s^2+4} \quad ; \quad Y(s) = \frac{2}{s^3-4s} = \frac{2}{s(s^2-4)}$$

$$* \ \frac{1}{s(s-2)(s+2)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+2}$$

$s=0$, $A = -\frac{1}{4}$; $s=2$, $B = \frac{1}{8}$; $C = \frac{1}{8}$

$$= \frac{-1}{4s} + \frac{1}{8(s-2)} + \frac{1}{8(s+2)}$$

$$y(x) = \ell^{-1}\{Y(s)\} = 2\ell^{-1}\left\{\frac{-1}{4s} + \frac{1}{8(s-2)} + \frac{1}{8(s+2)}\right\}$$

$$= \boxed{2\left(-\frac{1}{4} + \frac{1}{8}e^{2x} + \frac{1}{8}e^{-2x}\right)}$$

QUESTION 3. (10 points) Solve for $y(x)$: Given $y' + 6y = U(x-2)e^{3x}$; $y(0)=0$

$g(x)$

$sY(s) - \cancel{y(0)} + 6Y(s) = e^{-2s} \cdot e^{6} \cdot \frac{1}{s-3}$

$\mathcal{L}\{e^{3(x+2)}\} = \mathcal{L}\{e^{3x} e^{6}\} = e^{6}\mathcal{L}\{e^{3x}\} = e^{6} \cdot \frac{1}{s-3}$

$Y(s)(s+6) = \frac{e^{-2s} \cdot e^{6}}{s-3}$

$Y(s) = \frac{e^{-2s} \cdot e^{6}}{(s-3)(s+6)}$

$f(x) = \mathcal{L}^{-1}\left\{\frac{1}{(s-3)(s+6)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{9(s-3)} + \frac{-1}{9(s+6)}\right\} = \frac{1}{9}e^{3x} - \frac{1}{9}e^{-6x}$

$\frac{1}{(s-3)(s+6)} = \frac{A}{s-3} + \frac{B}{s+6}$, $s=3$, $s=-$, $A = \frac{1}{9}$, $B=$

$y(x) = \mathcal{L}^{-1}\{Y(s)\} = f(x-2)\,U(x-2) = e^{6}\left(\frac{1}{9}e^{3(x-2)} - \frac{1}{9}e^{-6(x-2)}\right) \cdot U(x-2)$

QUESTION 4. (12 points) Solve for $y(t)$ and $x(t)$: Given $y'(t) + 2x(t) = 0$ and $-2y(t) + x'(t) = 2$, where $x(0) = 0$, $y(0) = 0$.

Faculty information

Ayman Badawi, Department of Mathematics & Statistics, American University of Sharjah, P.O. Box 26666, Sharjah, United Arab Emirates.
E-mail: abadawi@aus.edu, www.ayman-badawi.com

~~$y'(t) + 2x(t) = 0$~~

$2x(t) + y'(t) = 0$

$x'(t) - 2y(t) = 2$, Apply laplace ,

$2X(s) + sY(s) - \cancel{y(0)} = 0$

$sX(s) - \cancel{x(0)} - 2Y(s) = \frac{2}{s}$

$X(s) = \frac{\det\begin{bmatrix} 0 & s \\ \frac{2}{s} & -2 \end{bmatrix}}{\det\begin{bmatrix} 2 & s \\ s & -2 \end{bmatrix}} = \frac{0-2}{-4-s^2} = \frac{-2}{-4-s^2} = \frac{2}{s^2+4}$

$x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \boxed{\sin 2t}$

substitute to find $y(t)$:

$x(t) = \sin 2t$

$x'(t) = 2\cos 2t$

$x'(t) - 2y(t) = 2$

$2\cos 2t - 2y(t) = 2$

$2y(t) = 2\cos 2t - 2$

$\boxed{y(t) = \cos 2t - 1}$

Name Doa'a Nassar, ID 43631

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MTH 205, Differential Equations , Exam Two

Ayman Badawi

QUESTION 1. (i) **(10 points)** Find the general solution to : $y^{(4)} + 4y^{(2)} = 0$

For y_h: $y = e^{\alpha x}$ $\quad C(\alpha) = \alpha^2(\alpha^2 + 4) = 0$

$\alpha_1 = 0 \ , \ \alpha_2 = 0 \ , \ \alpha_3 = 2i \ , \ \alpha_4 = -2i$

$y_1 = 1 \ , \ y_2 = 1x = x \ , \ y_3 = \cos(2x), \ y_4 = \sin(2x)$

$y_g = y_h = C_1 + C_2 x + C_3\cos(2x) + C_4 \sin(2x)$

(ii) **(10 points)** Find the general solution to : $y^{(4)} + 4y^{(2)} = 4x + 9$ [You may use (1) above]

$y_h = C_1 + C_2 x + C_3 \cos(2x) + C_4 \sin(2x)$

$y_p = (ax + b)x^2 = ax^3 + bx^2$

$y_p' = 3ax^2 + 2bx$

$y_p'' = 6ax + 2b$

$y_p^{(3)} = 6a$

$y_p^{(4)} = 0$

$0 + 4(6ax + 2b) = 4x + 9$

$24ax + 8b = 4x + 9$

$24ax = 4x \qquad 8b = 9$

$24a = 4 \qquad \boxed{b = 9/8}$

$\boxed{a = 1/6}$

$y_g = C_1 + C_2 x + C_3\cos(2x) + C_4 \sin(2x) + \frac{1}{6}x^3 + \frac{9}{8}x^2$

Name Yassin Habib, ID 30532 @ 10 a.m

Exam ONE, MTH 205, SPRING 2009

Ayman Badawi

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QUESTION 1. (25 points) 1)Find $\ell\{2^{3x+2}\}$

$$\Rightarrow \ell\left\{e^{(3x+2)\ln 2}\right\} = \left\{e^{3x\ln 2 + 2\ln 2}\right\} = \left\{e^{3x\ln 2} \cdot e^{2\ln 2}\right\}$$

$$= e^{2\ln 2}\,\ell\left\{e^{3\ln 2 x}\right\} = \boxed{\frac{e^{2\ln 2}}{s - 3\ln 2}} = 4\,\frac{1}{s-3\ln(2)}$$

2) Find $\ell\{x^2U(x-2)\}$

$(a=2)$

e^{-2s} $\Rightarrow g(x+2) = (x+2)^2$

$$\Rightarrow \ell\{g(x+2)\} = \ell\{(x+2)^2\} = \ell\{x^2+4x+4\} = \frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}$$

$$\Rightarrow e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right)$$

3) USE convolution to find $\ell^{-1}\{\frac{1}{s^2-s}\}$

$$= \ell^{-1}\left\{\frac{1}{s(s-1)}\right\} = \left\{\underbrace{\frac{1}{s}}_{F(s)} \cdot \underbrace{\frac{1}{s-1}}_{G(s)}\right\}$$

$$= 1 * e^x$$

$$= \int_0^x e^r\,dr$$

$$\Rightarrow e^r \Big|_{r=0}^{r=x} = e^x - e^0 = \boxed{e^x - 1}$$

4)Find $\ell^{-1}\{\frac{3e^{-2s}}{(s-4)^2+9}\}$, $f(x-2)\,u(x-2)$

$\Rightarrow \ell^{-1}\left\{e^{-2s}\cdot\underbrace{\frac{3}{(s-4)^2+9}}_{F(s)}\right\}$ $\Rightarrow$ $u(x-2)\sin(3x-6)\,e^{4x-8}$

$\Rightarrow f(x) = \sin 3x\, e^{4x}$

$\Rightarrow f(x-2) = \sin(3x-6)\, e^{4x-8}$

5) Find $\ell^{-1}\{\frac{s+5}{(2s+6)^3}\}$

$= \ell^{-1}\left\{\frac{s+5}{[2(s+3)]^3}\right\} = {}^{-1}\left\{\frac{s+5}{8(s+3)^3}\right\} = \frac{1}{8}\ell^{-1}\left\{\frac{s+5-2+2}{(s+3)^3}\right\} = \frac{1}{8}\ell^{-1}\left(\frac{1}{(s+3)^2}+\frac{2}{(s+3}\right.$

$= \frac{1}{8}\left(xe^{-3x} + x^2e^{-3x}\right) = \frac{xe^{-3x}}{8}(1+x)$

QUESTION 2. (10 points) Solve $y^{(2)} + 9y = 3, y(0) = y'(0) = 0$

Apply laplace

$\Rightarrow s^2Y_{(s)} - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0} + 9Y_{(s)} = \frac{3}{s}$

$\Rightarrow Y_{(s)}(s^2+9) = \frac{3}{s}$

$\Rightarrow Y_{(s)} = \frac{3}{s(s^2+9)}$

$\Rightarrow y_{(x)} = \ell^{-1}\left\{\frac{3}{s(s^2+9)}\right\}$

$= \ell^{-1}\left\{\frac{1}{s}\cdot\frac{3}{s^2+9}\right\}$

$= 1 * \sin 3x$

$\Rightarrow 1 * \sin 3x = \int_0^x \sin 3r\, dr$

$= -\frac{1}{3}\cos 3r \Big|_{r=0}^{r=x}$

$= -\frac{1}{3}(\cos 3x - 1)$

$\Rightarrow y(x) = \frac{1}{3} - \frac{\cos 3x}{3}$

QUESTION 3. (15 points) Solve for y(x), $y'(x) = e^{2x} - \int_0^x e^{2x-2r} y(r)\,dr,\ y(0) = 0$

$\Rightarrow y'(x) = e^{2x} - \underbrace{\int_0^x y(r)\, e^{2(x-r)}\, dr}_{\to\ y(x) * e^{2x}}$

$\mathcal{L}\{y(x) * e^{2x}\} = \dfrac{Y_{(s)}}{s-2}$

Apply laplace

$\Rightarrow sY_{(s)} - \cancel{y(0)}^{\,0} = \dfrac{1}{s-2} - \mathcal{L}\{y(x) * e^{2x}\}$

$\Rightarrow sY_{(s)} = \dfrac{1}{s-2} - \dfrac{Y_{(s)}}{s-2}$

$\Rightarrow sY_{(s)} + \dfrac{Y_{(s)}}{s-2} = \dfrac{1}{s-2}$

$\Rightarrow Y_{(s)}\left(s + \dfrac{1}{s-2}\right) = \dfrac{1}{s-2}$

$\Rightarrow Y_{(s)}\left(\dfrac{s^2-2s+1}{s-2}\right) = \dfrac{1}{s-2}$

$\Rightarrow Y_{(s)}\left(\dfrac{(s-1)^2}{s-2}\right) = \dfrac{1}{s-2}$

$\Rightarrow Y_{(s)} = \dfrac{1}{(s-1)^2}$

$\Rightarrow y(x) = \mathcal{L}^{-1}\left\{\dfrac{1}{(s-1)^2}\right\} = \underline{\underline{xe^x}}$

QUESTION 4. (15 points) Solve for $x(t)$ and $y(t)$ if $x'(t) - 0.5y(t) = t$ and $x(t) + \int_0^t y(r)\,dr = 2t^2, x(0) = y(0) = 0$

① $x'(t) - 0.5\,y(t) = t$ $\qquad x(0) = y(0) = 0.$

② $x(t) + \int_0^t y(r)\,dr = 2t^2$

Apply laplace

$\mathcal{L}\{①\}$

$\Rightarrow sX_{(s)} - \cancel{x(0)}^{\,0} - 0.5\,(Y_{(s)}) = \dfrac{1}{s^2}$

$\Rightarrow \boxed{sX_{(s)} - 0.5\,Y_{(s)} = \dfrac{1}{s^2}}$

$\mathcal{L}\{②\}$

$\Rightarrow \boxed{X_{(s)} + \dfrac{Y_{(s)}}{s} = \dfrac{4}{s^3}}$

$X_{(s)} = \dfrac{\det\begin{bmatrix} 1/s^2 & -0.5 \\ 4/s^3 & 1/s \end{bmatrix}}{\det\begin{bmatrix} s & -0.5 \\ 1 & 1/s \end{bmatrix}}$

$= \dfrac{\dfrac{1}{s^3} + \dfrac{2}{s^3}}{1 + 0.5} = \dfrac{3}{s^3} \times \dfrac{2}{3}$

$= \dfrac{2}{s^3}$

$x_{(t)} = \mathcal{L}^{-1}\left\{\dfrac{2}{s^3}\right\} = \boxed{t^2}$

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$$x'(t) - 0.5\,y(t) = t$$

$$x'(t) = 2t, \quad \Rightarrow \quad 2t - 0.5\,y(t) = t$$

$$\Rightarrow \quad -\frac{1}{2}\,y(t) = t - 2t$$

$$\Rightarrow y(t) = -2(t - 2t)$$

$$= -2t + 4t$$

$$= \underline{\underline{2t}}$$

QUESTION 5. (15 points) Find the general solution to $y^{(3)} + 2y^{(2)} + y' = 0$. USE THE HOMOGENEOUS METHOD.

Solution: $y = e^{mx}$

char(D.E) $\Rightarrow m^3 + 2m^2 + m$

set char(D.E) = 0, $\Rightarrow m(m^2 + 2m + 1) = 0$

$\Rightarrow m(m+1)(m+1) = 0$

$\Rightarrow m = 0, \; m = -1, \; m = -1$

$\downarrow e^0 = 1$ $\quad \downarrow e^{-x}$ $\quad \downarrow xe^{-x}$

$\Rightarrow y_g = c_1 + c_2 e^{-x} + c_3 x e^{-x}$

QUESTION 6. (20 points) a) Find the general solution to $y^{(2)} + 16y = 0$

Solution: $y = e^{mx}$

char(D.E): $m^2 + 16$

set char D.E = 0: $m^2 + 16 = 0$

$\Rightarrow m^2 = -16$

$\Rightarrow m = \pm\sqrt{-16}$

$\Rightarrow$ imaginary $\Rightarrow$ use one solution

$+\sqrt{-16} = a + bi$

$\Rightarrow \sqrt{-16} = a + bi$

$\Rightarrow a = 0, \; b = \sqrt{16} = 4$

$\Rightarrow y_1 = e^{ax} \cos bx$

$= \cos 4x$

$y_2 = e^{ax} \sin bx$

$= \sin 4x$

$\Rightarrow y_G = c_1 \cos 4x + c_2 \sin 4x$

b) If $y(0) = 0$ and $y'(\pi/8) = 0$, what will be the solution of the D.E in part (a)? Does that contradict one of the Theorem in the book? EXPLAIN (Note $sin(\pi/2) = 1, cos(\pi/2) = 0, sin(0) = 0, cos(0) = 1$)

$y(x) = c_1 \cos 4x + c_2 \sin 4x$

$\Rightarrow y(0) = 0 = c_1$

$\Rightarrow c_1 = 0$

$y'(\frac{\pi}{8}) = -4c_1 \sin 4x + 4c_2 \cos 4x$

$0 = -4c_1$

$\Rightarrow c_1 = 0$

$y_a = c_2 \sin 4x$

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c) If $y(0) = 0$ and $y'(\pi/8) = 1$, what will be the solution of the D.E. in part (a)? Does that contradict one of the Theorem in the book? EXPLAIN

* has to be same x value.

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Faculty information

Ayman Badawi, Department of Mathematics & Statistics, American University of Sharjah, P.O. Box 26666, Sharjah, United Arab Emirates.
E-mail: abadawi@aus.edu, www.ayman-badawi.com

b)

$y^2 + 16y = 0$

Apply laplace $\Rightarrow s^2 Y_{(s)} - s y(0) - y'(0) + 16 Y_{(s)} = 0$

$\Rightarrow Y_{(s)} [s^2 + 16] = s y(0) + y'(0)$

$\Rightarrow Y_{(s)} = \dfrac{s y(0) + y'(0)}{s^2 + 16}$

$\Rightarrow Y_{(s)} = \dfrac{s c_1}{s^2+16} + \dfrac{c_2}{s^2+16}$

$= c_1 \cos 4x + c_2 \sin 4x$

b) $y(0) = 0$

$y'(\pi/8) = 0$.

$\Rightarrow Y_{(s)} = \dfrac{y'(0)}{s^2 + 16}$

$\Rightarrow y(x) = \dfrac{c_1}{4} \sin 4x$

$y'(x) = c_1 \cos 4x$

$y'(\frac{\pi}{8}) = c_1 (0)$

$= 0$

$\Rightarrow$ no contradiction

c) $y'(\frac{\pi}{8}) = 0$, can't be $= 1$

there is a contradiction since $1 = c(0)$

$\Rightarrow c = \dfrac{1}{0}$ $\Rightarrow$ no constant can be multiplied to satisfy the theorm

$\Rightarrow$ contradiction.

ok

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